Eigenfunctions of $GL(N, \mathbb{R})$ Toda chain:
The Mellin-Barnes representation.

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Abstract

The recurrent relations between the eigenfunctions for $GL(N, \mathbb{R})$ and $GL(N-1, \mathbb{R})$ quantum Toda chains is derived. As a corollary, the Mellin-Barnes integral representation for the eigenfunctions of a quantum open Toda chain is constructed for the N-particle case.

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1 Introduction

Recently a new method of construction of the eigenfunctions for the periodic Toda chain have been introduced [1]. It relies on generalized Fourier transform expansion over eigenfunctions of an open Toda chain which coincide with the Whittaker functions of the $GL(N-1, \mathbb{R})$ group [2]-[4].

In the original papers [2]-[4] the Whittaker functions were constructed by purely algebraic methods in terms of the Iwasawa decomposition for the corresponding group. But it turns out that there is an alternative way to construct these functions directly on the level of R-matrix formalism using essentially the integrable properties of the model.

This paper was inspired by one remark of E. Sklyanin, that integral formula for the eigenfunction of periodic Toda chain [1] can be used to obtain some recurrent relations for N-particles eigenfunctions of open Toda chain through the $N-1$ particles ones. Sklyanin's motivations based on a possibility to introduce a formal parameter to the Baxter equation [5] and then, putting it equal to zero, to transform the second order difference equation to the first order one. In our paper we find rigorous proof of this observation which is free of the limiting procedure and find an universal way to construct "auxiliary" eigenfunctions staying completely in the framework of the R-matrix formalism without any reference to an algebraic scheme developed in [2]-[4]. To compare with the well known results on eigenfunctions of open Toda chain [6, 7], our approach presents not only a new integral representation for the eigenfunctions but, together with generalized Fourier transformation, it gives a self-consistent method to solve the spectral problem for periodic Toda chain. We hope that this method can be applied for some other classes of integrable systems.

2 The model

We start with the R-matrix formalism for the quantum periodic Toda chain [5]. Let

$$L_n(\lambda) \;=\; \begin{pmatrix} \lambda - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix} \tag{2.1}$$

be the corresponding Lax operator where $[x_n, p_m] = i\hbar \delta_{nm}$. The N-particle monodromy matrix

$$T_N(\lambda) \;\stackrel{\mathrm{def}}{=}\; L_N(\lambda) \dots L_1(\lambda) \;\equiv\; \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix} \tag{2.2}$$

satisfies the standard RTT relations with the rational R-matrix. In particular,

$$(\lambda - \mu + i\hbar)A_N(\mu)C_N(\lambda) \;=\; (\lambda - \mu)C_N(\lambda)A_N(\mu) + i\hbar A_N(\lambda)C_N(\mu) \tag{2.3}$$

The eigenfunctions for the periodic spectral problem have been constructed [1] with the help of Weyl invariant function $\psi_{\gamma_1, \dots, \gamma_{N-1}}(x_1, \dots, x_{N-1}) \equiv \psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$ which is fast decreasing in the

regions $x_k \gg x_{k+1}$, $(k = 1, \ldots, N-1)$ and satisfies to equations

$$C_N(\lambda)\psi_{\boldsymbol{\gamma}} = -\, e^{x_N} \prod_{m=1}^{N-1} (\lambda - \gamma_m)\, \psi_{\boldsymbol{\gamma}} \tag{2.4}$$

$$A_N(\gamma_j)\psi_{\boldsymbol{\gamma}} = i^{-N} e^{-x_N} \psi_{\boldsymbol{\gamma} - i\hbar \boldsymbol{e}_j} \qquad (j = 1, \ldots, N-1) \tag{2.5}$$

where $\boldsymbol{e}_j$ is j-th basis vector in $\mathbb{R}^{N-1}$. It is easy to see that the equation (2.5) is compatible with (2.4) due to commutation relation (2.3).

Actually, the function $\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$ is an appropriate solution for $N-1$ particle open Toda chain. Indeed, the operator $A_{N-1}(\lambda)$ (arising from the $N-1$ particle problem) is the generating function for the Hamiltonians of the $GL(N-1, \mathbb{R})$ Toda chain. Using the obvious relations between the elements of the monodromy matrices $T_N(\lambda)$ and $T_{N-1}(\lambda)$:

$$A_N(\lambda) = (\lambda - p_N)A_{N-1}(\lambda) + e^{-x_N} C_{N-1}(\lambda)$$
$$C_N(\lambda) = -e^{x_N} A_{N-1}(\lambda) \tag{2.6}$$

the equations (2.4) and (2.5) can be written in the equivalent form

$$A_{N-1}(\lambda)\psi_{\boldsymbol{\gamma}} = \prod_{m=1}^{N-1} (\lambda - \gamma_m)\, \psi_{\boldsymbol{\gamma}} \tag{2.7}$$

$$C_{N-1}(\gamma_j)\psi_{\boldsymbol{\gamma}} = i^{-N} \psi_{\boldsymbol{\gamma} - i\hbar \boldsymbol{e}_j} \qquad (j = 1, \ldots, N-1) \tag{2.8}$$

These equations fix (up to $i\hbar$-periodic common factor) the Weyl invariant Whittaker function for $GL(N-1, \mathbb{R})$ group. In [1] we choose the factor in such a way that $\psi_{\boldsymbol{\gamma}}$ is an entire function in $\boldsymbol{\gamma}$ and the following asymptotics hold:

$$\psi_{\boldsymbol{\gamma}} \sim |\gamma_j|^{\frac{2-N}{2}} \exp\left\{ -\frac{\pi}{2\hbar}(N-2)|\gamma_j| \right\} \tag{2.9}$$

as $|\operatorname{Re}\gamma_j| \to \infty$ in the finite strip of complex plane.

Remark 2.1 *The function $C_{N-1}(\lambda)\psi_{\boldsymbol{\gamma}}$ is a polynomial in λ of order $N-2$. Therefore, this polynomial is restored by their $N-1$ values at given points $\gamma_1, \ldots, \gamma_{N-1}$. Hence, one obtains the interpolation formula*

$$C_{N-1}(\lambda)\psi_{\boldsymbol{\gamma}} = i^{-N} \sum_{j=1}^{N-1} \psi_{\boldsymbol{\gamma} - i\hbar \boldsymbol{e}_j} \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \tag{2.10}$$

Let us introduce the key object - the *auxiliary* function

$$\Psi_{\boldsymbol{\gamma}, \epsilon}(x_1, \ldots, x_N) \overset{\text{def}}{=} e^{\frac{i}{\hbar}\left(\epsilon - \sum\limits_{m=1}^{N-1} \gamma_m\right)x_N} \psi_{\boldsymbol{\gamma}}(\boldsymbol{x}) \tag{2.11}$$

where ϵ is an arbitrary parameter. From (2.7), (2.10), and (2.6) it is readily seen that this function satisfies to equations

$$A_N(\lambda)\Psi_{\boldsymbol{\gamma},\epsilon} = \left(\lambda - \epsilon + \sum_{m=1}^{N-1}\gamma_m\right)\prod_{j=1}^{N-1}(\lambda - \gamma_j)\,\Psi_{\boldsymbol{\gamma},\epsilon} + i^{-N}\sum_{j=1}^{N-1}\Psi_{\boldsymbol{\gamma}-i\hbar\boldsymbol{e}_j,\epsilon}\prod_{m\neq j}\frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \quad (2.12)$$

$$C_N(\lambda)\Psi_{\boldsymbol{\gamma},\epsilon} = -e^{x_N}\prod_{j=1}^{N-1}(\lambda - \gamma_j)\,\Psi_{\boldsymbol{\gamma},\epsilon} \quad (2.13)$$

3 The problem

Let the Weyl invariant Whittaker function $\psi_{\gamma_1,\dots,\gamma_{N-1}}(x_1,\dots,x_{N-1})$ for $GL(N-1,\mathbb{R})$ Toda chain is given. The problem is to find the corresponding solution for $GL(N,\mathbb{R})$ Toda chain using the above information, i.e. to construct the Weyl invariant Whittaker function $\psi_{\lambda_1,\dots,\lambda_N}(x_1,\dots,x_N)$ satisfying to equations

$$A_N(\lambda)\psi_{\lambda_1,\dots,\lambda_N} = \prod_{k=1}^{N}(\lambda - \lambda_k)\,\psi_{\lambda_1,\dots,\lambda_N} \quad (3.1a)$$

$$C_N(\lambda_n)\psi_{\lambda_1,\dots,\lambda_N} = i^{-N-1}\,\psi_{\lambda_1,\dots,\lambda_n-i\hbar,\dots,\lambda_N} \quad (n = 1,\dots,N) \quad (3.1b)$$

and obeying the asymptotics

$$\psi_{\lambda_1,\dots,\lambda_N} \sim |\lambda_n|^{\frac{1-N}{2}}\exp\left\{-\frac{\pi}{2\hbar}(N-1)|\lambda_n|\right\} \quad (3.2)$$

as $|\operatorname{Re}\lambda_n| \to \infty$, in terms of the function $\psi_{\boldsymbol{\gamma}}(\boldsymbol{x})$. It is clear that the action of the operators $A_N(\lambda)$ and $C_N(\lambda)$ is nicely defined when acting on the auxiliary functions $\Psi_{\boldsymbol{\gamma},\epsilon}(x_1,\dots,x_N)$. Therefore, it is reasonable to assume that the solution for $GL(N,\mathbb{R})$ Toda chain is described by appropriate (generalized) Fourier transformation of the function $\Psi_{\boldsymbol{\gamma},\epsilon}(x_1,\dots,x_N)$. This is in complete analogy with the corresponding construction for the periodic case [1].

4 Main statements

Theorem 4.1 *Let $\Psi_{\boldsymbol{\gamma},\epsilon}(\boldsymbol{x}, x_N)$ be the auxiliary function for N-periodic Toda chain, i.e. it is defined in terms of the Weyl invariant Whittaker function for $GL(N-1,\mathbb{R})$ Toda chain according to (2.11). Let $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_N) \in \mathbb{C}^N$ be the set of indeterminates. Let*

$$\mu(\boldsymbol{\gamma}) = (2\pi\hbar)^{N-1}(N-1)!\prod_{j<k}\left|\Gamma\left(\frac{\gamma_j - \gamma_k}{i\hbar}\right)\right|^2 \quad (4.1)$$

$$Q(\gamma_1,\dots,\gamma_{N-1}|\lambda_1,\dots,\lambda_N) = \prod_{j=1}^{N-1}\prod_{k=1}^{N}h^{\frac{\gamma_j-\lambda_k}{i\hbar}}\Gamma\left(\frac{\gamma_j-\lambda_k}{i\hbar}\right) \quad (4.2)$$

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Then the Weyl invariant Whittaker function for $GL(N, \mathbb{R})$ Toda chain is given by recurrent formula

$$\psi_{\lambda_1, \ldots, \lambda_N}(x_1, \ldots, x_N) = \int_{\mathcal{C}} \mu^{-1}(\boldsymbol{\gamma}) Q(\boldsymbol{\gamma}; \boldsymbol{\lambda}) \Psi_{\boldsymbol{\gamma}; \lambda_1 + \ldots + \lambda_N}(x_1, \ldots, x_N) \, d\boldsymbol{\gamma} \qquad (4.3)$$

where the integration is performed along the horizontal lines with $\operatorname{Im} \gamma_j > \max_k \{\operatorname{Im} \lambda_k\}$.

5   Proof of the Theorem

First of all, the integral (4.3) is correctly defined. Indeed, the function

$$q(\gamma | \lambda_1, \ldots, \lambda_N) \equiv \prod_{k=1}^{N} h^{\frac{\gamma - \lambda_k}{i\hbar}} \Gamma\left(\frac{\gamma - \lambda_k}{i\hbar}\right) \qquad (5.1)$$

obeys the asymptotics

$$q(\gamma; \boldsymbol{\lambda}) \sim |\gamma|^{-N/2} \exp\left\{ -\frac{\pi N}{2\hbar} |\gamma| \right\} \qquad (5.2)$$

as $|\operatorname{Re} \gamma| \to \infty$ in finite horizontal strip while

$$\mu^{-1}(\boldsymbol{\gamma}) \sim |\gamma_j|^{N-2} \exp\left\{ \frac{\pi}{\hbar}(N-2) |\gamma_j| \right\} \qquad (5.3)$$

Hence, the integral in (4.3) is absolutely convergent due to asymptotics (2.9).

Let us verify that the relation (3.1a) holds. Using (2.12) one finds

$$A_N(\lambda) \psi_{\lambda_1, \ldots, \lambda_N}(x_1, \ldots, x_N) =$$

$$= \int_{\mathcal{C}} \left(\lambda - \sum_{k=1}^{N} \lambda_k + \sum_{m=1}^{N-1} \gamma_m \right) \prod_{j=1}^{N-1} (\lambda - \gamma_j) \, \mu^{-1}(\boldsymbol{\gamma}) Q(\boldsymbol{\gamma}; \boldsymbol{\lambda}) \, \Psi_{\boldsymbol{\gamma}, \lambda_1 + \ldots + \lambda_N}(x_1, \ldots, x_N) \, d\boldsymbol{\gamma} +$$

$$+ i^{-N} \sum_{j=1}^{N-1} \int_{\mathcal{C}} \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \, \mu^{-1}(\boldsymbol{\gamma}) Q(\boldsymbol{\gamma}; \boldsymbol{\lambda}) \, \Psi_{\boldsymbol{\gamma} - i\hbar \boldsymbol{e}_j, \lambda_1 + \ldots + \lambda_N}(x_1, \ldots, x_N) \, d\boldsymbol{\gamma} \qquad (5.4)$$

We shift the appropriate integrations $\gamma_j \to \gamma_j + i\hbar$ and use the functional equation

$$\mu^{-1}(\boldsymbol{\gamma} + i\hbar \boldsymbol{\delta}_j) = (-1)^N \mu^{-1}(\boldsymbol{\gamma}) \prod_{m \neq j} \frac{\gamma_j - \gamma_m + i\hbar}{\gamma_j - \gamma_m} \qquad (5.5)$$

The second integrand in (5.4) has no poles in any finite horizontal strip in the upper half-plane $\operatorname{Im} \gamma_j > \max_k \{\operatorname{Im} \lambda_k\}$ (all possible poles are cancelled by appropriate zeros of the function $\mu^{-1}(\boldsymbol{\gamma})$). Moreover, the intergand is fast decreasing in this strip as $\operatorname{Re} \gamma_j \to \pm\infty$.

As a consequence, the integral over the strip vanishes. Therefore, it is possible to deform the shifted contour to the original one. Hence, one arrives at the relation

$$A_N(\lambda)\psi_{\lambda_1,\ldots,\lambda_N} = \int_{\mathcal{C}} \left\{ \left(\lambda - \sum_{k=1}^{N}\lambda_k + \sum_{m=1}^{N-1}\gamma_m \right) \prod_{j=1}^{N-1}(\lambda-\gamma_j)Q(\boldsymbol{\gamma};\boldsymbol{\lambda}) + \right.$$

$$\left. + i^N \sum_{j=1}^{N-1} \prod_{m\neq j} \frac{\lambda-\gamma_m}{\gamma_j-\gamma_m} \, Q(\boldsymbol{\gamma}+i\hbar\boldsymbol{e}_j;\boldsymbol{\lambda}) \right\} \mu^{-1}(\boldsymbol{\gamma})\Psi_{\boldsymbol{\gamma},\lambda_1+\ldots+\lambda_N}d\boldsymbol{\gamma} \tag{5.6}$$

The function $Q(\boldsymbol{\gamma};\boldsymbol{\lambda})$ satisfies to equations

$$\prod_{k=1}^{N}(\gamma_j - \lambda_k)\,Q(\boldsymbol{\gamma};\boldsymbol{\lambda}) = i^N Q(\boldsymbol{\gamma}+i\hbar\boldsymbol{e}_j;\boldsymbol{\lambda}) \tag{5.7}$$

for any $j = 1,\ldots,N-1$. Therefore, the relation (5.6) acquires the form

$$A(\lambda)\psi_{\lambda_1,\ldots,\lambda_N} = \int_{\mathcal{C}} \left\{ \left(\lambda - \sum_{k=1}^{N}\lambda_k + \sum_{m=1}^{N-1}\gamma_s \right) \prod_{j=1}^{N-1}(\lambda-\gamma_j) + \right.$$

$$\left. + \sum_{j=1}^{N-1}\prod_{k=1}^{N}(\gamma_j - \lambda_k) \prod_{m\neq j} \frac{\lambda-\gamma_m}{\gamma_j-\gamma_m} \right\} \mu^{-1}(\boldsymbol{\gamma})Q(\boldsymbol{\gamma};\boldsymbol{\lambda})\Psi_{\boldsymbol{\gamma},\lambda_1+\ldots+\lambda_N}d\boldsymbol{\gamma} \tag{5.8}$$

But the expression in curly brackets is nothing but the polynomial $\prod_{k=1}^{N}(\lambda-\lambda_k)$. Indeed, any polynomial with the leading terms $F(\lambda) = \lambda^N + f_1\lambda^{N-1} + \ldots$ can be uniquely restored by its values at any $N-1$ arbitrary points $\gamma_1,\ldots,\gamma_{N-1}$ according to interpolation formula

$$F(\lambda) = \left(\lambda + f_1 + \sum_{m=1}^{N-1}\gamma_m \right) \prod_{j=1}^{N-1}(\lambda - \gamma_j) + \sum_{j=1}^{N-1} F(\gamma_j) \prod_{m\neq j} \frac{\lambda-\gamma_m}{\gamma_j-\gamma_m} \tag{5.9}$$

- in our case

$$F(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k) \tag{5.10}$$

with $f_1 = -\lambda_1 - \ldots - \lambda_N$. Hence, we obtain (3.1a).

Further, we consider the relation (3.1b). Using (2.13) one obtains

$$C_N(\lambda_n)\psi_{\lambda_1,\ldots,\lambda_N} = -e^{x_N}\int_{\mathcal{C}} \mu^{-1}(\boldsymbol{\gamma})Q(\boldsymbol{\gamma};\boldsymbol{\lambda}) \prod_{j=1}^{N-1}(\lambda_n - \gamma_j)\Psi_{\boldsymbol{\gamma},\lambda_1+\ldots+\lambda_N}\,d\boldsymbol{\gamma} \tag{5.11}$$

Clearly,

$$e^{x_N}\Psi_{\boldsymbol{\gamma},\lambda_1+\ldots+\lambda_N} = \Psi_{\boldsymbol{\gamma},\lambda_1+\ldots+\lambda_N-i\hbar} \tag{5.12}$$

and, therefore, the relation (5.11) acquires the form

$$C_N(\lambda_n)\psi_{\lambda_1,\dots,\lambda_N} = (-1)^N \int_{\mathcal{C}} \mu^{-1}(\boldsymbol{\gamma})Q(\boldsymbol{\gamma};\boldsymbol{\lambda}) \prod_{j=1}^{N-1}(\gamma_j - \lambda_n)\Psi_{\boldsymbol{\gamma},\lambda_1+\dots+\lambda_N-i\hbar}\, d\boldsymbol{\gamma} \qquad (5.13)$$

Evidently, the function $Q(\boldsymbol{\gamma};\boldsymbol{\lambda})$ satisfies to equation

$$\prod_{j=1}^{N-1}(\gamma_j - \lambda_n)Q(\boldsymbol{\gamma};\boldsymbol{\lambda}) = i^{N-1}Q(\boldsymbol{\gamma};\boldsymbol{\lambda} - i\hbar\boldsymbol{e}_n) \qquad (5.14)$$

Hence, we prove that function (4.3) obeys the relations (3.1b).

The final step is to prove that the function (4.3) is the genuine Whittaker function. The integrand in (4.3) decreases exponentially as $\gamma_j \to -i\infty$, $(j = 1,\dots,N-1)$ and, as consequence, the integrals over large semi-circles in the lower half-plane vanish. Using the Cauchy formula to calculate the integral (4.3) in the asymptotic region $x_{k+1} \gg x_k$, $(k = 1,\dots,N-1)$, it is easy to see that the asymptotics of the function $\psi_{\lambda_1,\dots,\lambda_N}$ are determined precisely in terms of the corresponding Harish-Chandra functions (see, for example, [4]):

$$\psi_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \sum_{s\in W} \hbar^{-2i(s\boldsymbol{\lambda},\boldsymbol{\rho})/\hbar} \prod_{j<k}\Gamma\left(\frac{s\lambda_j - s\lambda_k}{i\hbar}\right) e^{\frac{i}{\hbar}(s\boldsymbol{\lambda},\boldsymbol{x})} + O\left(\max\left\{e^{x_k - x_{k+1}}\right\}_{k=1}^{N-1}\right) \qquad (5.15)$$

(in the last formula the summation is performed over the Weyl group). Hence, we construct exactly the Weyl invariant Whittaker function. Moreover, using the Stirling formula for the Γ-functions, it is easy to see that the asymptotics (3.2) hold. Theorem is proved. ∎

6   The Mellin-Barnes representation

Theorem 6.1 *Let a set $||\gamma_{jk}||$ be the lower triangular $N \times N$ matrix. The solution to eqs.(3.1) can be written (up to inessential numerical factor) in the form of multiple Mellin-Barnes integrals:*

$$\psi_{\gamma_{N1},\dots,\gamma_{NN}}(x_1,\dots,x_N) =$$

$$= \int_{\mathcal{C}} \prod_{n=1}^{N-1} \frac{\prod_{j=1}^{n}\prod_{k=1}^{n+1} \hbar^{\frac{\gamma_{nj}-\gamma_{n+1,k}}{i\hbar}}\Gamma\left(\frac{\gamma_{nj}-\gamma_{n+1,k}}{i\hbar}\right)}{\prod_{\substack{j,k=1\\j<k}}^{n}\left|\Gamma\left(\frac{\gamma_{nj}-\gamma_{nk}}{i\hbar}\right)\right|^2} \exp\left\{\frac{i}{\hbar}\sum_{n,k=1}^{N} x_n\left(\gamma_{nk}-\gamma_{n-1,k}\right)\right\} \prod_{\substack{j,k=1\\j\leq k}}^{N-1} d\gamma_{jk}$$

$$(6.1)$$

where the integral should be understand as follows: first we integrate on γ_{11} over the line $\operatorname{Im}\gamma_{11} > \max\{\operatorname{Im}\gamma_{21},\operatorname{Im}\gamma_{22}\}$; then we integrate on the set $(\gamma_{21},\gamma_{22})$ over the lines $\operatorname{Im}\gamma_{2j} > \max_m\{\operatorname{Im}\gamma_{3m}\}$ and so on. The last integrations should be performed on the set of variables $(\gamma_{N-1,1}\dots,\gamma_{N-1,N-1})$ over the lines $\operatorname{Im}\gamma_{N-1,k} > \max_m\{\operatorname{Im}\gamma_{N,m}\}$.

The proof is straightforward resolution of the recurrent relations (4.3) starting with trivial Whittaker function $\psi_{\gamma_{11}}(x_1) = \exp\{\frac{i}{\hbar}\gamma_{11}x_1\}$.

Acknowledgments

References

[1] S.Kharchev, D.Lebedev, *Integral representation for the eigenfunctions of quantum periodic Toda chain*, hep-th/9910265 (to be publ. in Lett.Math.Phys.)

[2] H.Jacquet, *Fonctions de Whittaker associées aux groupes de Chevalley*, Bull. Soc. Math. France, (1967), **95**, 243-309.

[3] G.Schiffmann, *Intégrales d'entrelacement et fonctions de Whittaker*, Bull.Soc.Math. France, (1971), **99**, 3-72.

[4] M.Hashizume *Whittaker models for real reductive groups*, J.Math.Soc. Japan, (1979), **5**, 394-401.
Whittaker functions on semisimple Lie groups, Hiroshima Math.J., (1982), **12**, 259-293.

[5] E.Sklyanin, *The quantum Toda chain*, Lect.Notes in Phys., (1985), **226**, 196-233.

[6] B.Kostant, *Quantization and representation theory* In: Representation theory of Lie Groups. Proc.SRC/LMS research Symp., Oxford 1977, London Math. Soc. Lecture Notes, *34*,287-316.

[7] M.Semenov-Tian-Shansky. *Quantum Toda lattices. Spectral theory and scattering.* Preprint LOMI R-3-84. Leningrad, 1984. 64p.
M.Semenov-Tian-Shansky, *Quantization of Open Toda Lattices.* Encyclopaedia of Mathematical Sciences, vol. 16. Dynamical Systems VII. Ch. 3. Springer Verlag, 1994, pp.226-259.